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Stochastic model of crack paths in composites based on the Langevin equation

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Abstract A stochastic model based on the Langevin equation was used to describe crack paths in composites. A single crack path in a fiberglass reinforced epoxy matrix composite was used to predict Langevin parameters quantifying architectural drift and material variability. The predicted Langevin parameters were consistent with moderate architectural drift (Langevin drift parameter of 0.04 ± 0.02) and moderate material variability (Langevin variability parameter of 0.05 ± 0.03). These Langevin parameters were then used to predict a family of crack paths exhibiting the same stochastic characteristics as the original crack path. Architectural drift was qualitatively related to the orientation of the reinforcing phase. A strong correlation between the predicted and experimental crack path suggests the utility of the Langevin model to quantify crack paths.

Modeling of crack paths as a function of architecture in composites has been the subject of previous research efforts, and the goal of these research efforts has been to optimize mechanical performance by optimizing the architecture. Crack paths in brittle matrix composites containing cylindrical particles were modeled using a linear elastic boundary element method [1]. Crack paths in particulate reinforced composites were modeled with an electrical analog [2]. The path of least electrical resistance was used as a model of the path of maximum strain energy release rate. Stochastic methods were used to model crack paths in a polycrystalline ceramic modeled as square array of grains [3]. Crack paths were described as a combination of transgranular and intergranular sections, and stochastic methods were used to determine of probability of each mode. Stochastic methods were also used to model failure of ceramic matrix composites containing crossplys [4]. While crack paths were not predicted in that study, the stochastic model did consider the affect of 90° crossplys on composite behavior. The effect of composite architecture on crack paths has been recently reported [5, 6].

The purpose of this communication is to report a stochastic model based on the Langevin equation [7] for a crack path in a fiberglass reinforced epoxy composite. The Langevin equation is a model of geometric Brownian motion, and has been used to model financial markets. The basic form of the Langevin equation given by:

$$dR = \mu R(c)dC + \sigma R(c)d\omega(c) \qquad (1)$$

where C = independent variable; R = dependent variable; μ = Langevin drift parameter; σ = Langevin variability parameter; dω(c) = Gaussian stochastic.

Equation 1 describes two simultaneous changes superimposed over one another to determine the combined effect on a dependent variable. The first term on the right-hand side of Eq. 1 is a deterministic term. The deterministic term is proportional to the current magnitude of the independent variable ($R(c)$), and the magnitude of change in the independent variable (dC). The proportionality constant of the deterministic term is given by μ, and is known as the Langevin drift parameter.

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The second term on the right-hand side of Eq. 1 is a stochastic term. The stochastic term is proportional to the current magnitude of the independent variable ($R(c)$), and the magnitude of a stochastic ($d\omega(c)$). The stochastic is a term randomly selected from a standard Gaussian distribution (mean of 0, and standard deviation of 1). The proportionality constant of the stochastic term is given by σ, and is known as the Langevin variability parameter. The superposition of both a deterministic change and a stochastic change in Eq. 1 is the basis of Brownian motion. The dependence of each change on the current magnitude of the independent variable is described as geometric Brownian motion.

Equation 1 can be solved to yield the probability distribution of R changing from R_1 to R_2 [7]:

$$p = \frac{1}{\sqrt{2\pi(\sigma R_2)^2(\Delta C)}} e^{-\frac{\left(\ln\left(\frac{R_2}{R_1}\right) - (\Delta C)\left(\mu - \frac{\sigma^2}{2}\right)\right)^2}{2\sigma^2 \Delta C}} \quad (2)$$

where R_1 = initial value of the dependent variable; R_2 = final value of the dependent variable; ΔC = change in magnitude of the independent variable.

Equation 2 is a log-normal probability distribution. The mean of a dependent variable (R_{mean}) undergoing change consistent with a log-normal distribution is given by:

$$R_{\text{mean}} = R_1 e^{\mu(\Delta C)} \quad (3)$$

The variance of a dependent variable (R_{var}) undergoing change consistent with a log-normal distribution is given by:

$$R_{\text{var}} = (R_1)^2 e^{2\mu(\Delta C)} \left\{ e^{\sigma^2(\Delta C)} - 1 \right\} \quad (4)$$

If changes are observed for period of time in a system described by Eqs. 1 and 2, both Langevin parameters (μ and σ) can be determined using Eqs. 3 and 4. It would then be possible to predict future changes in dependent variable (R) for known changes in the independent variable (C). This potential is the origin of the interest in the Langevin equation applied to financial markets.

Consider a two-dimensional composite described by a two-dimensional array of square material elements (Fig. 1), and each material element can be identified by the horizontal (C) and vertical (R) position of its center. When a crack propagates cross the two-dimensional composite, a trace of the crack tip position in R–C space results in a crack path. The crack path can be described by a two-dimensional plot of an independent variable (C) and a dependent variable (R). The crack path can then be model using Eq. 1, with the two Langevin parameters (μ and σ) determined from Eqs. 3 and 4. Using these two Langevin parameters, the crack path can be quantified. Using these Langevin parameters, a family of crack paths with the same stochastic descriptors as the original crack path can be predicted.

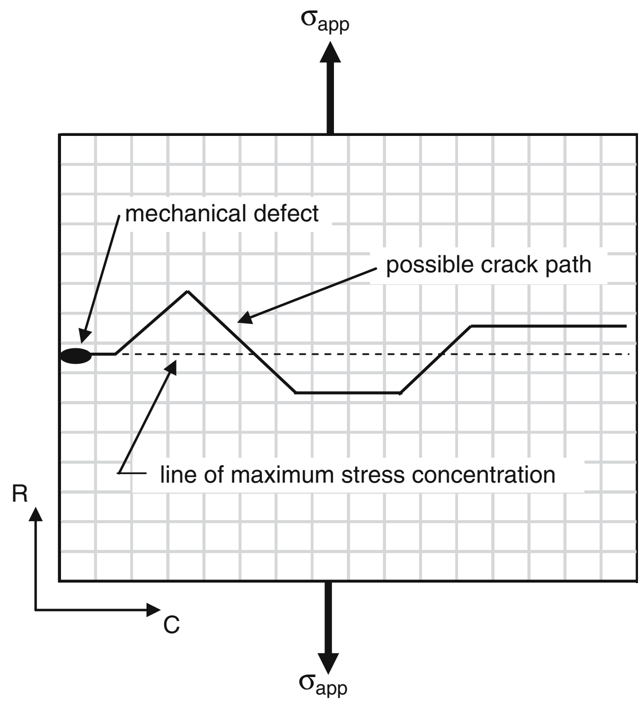

Fig. 1 Schematic representation of a crack path in a two-dimensional model with the crack path originating at a mechanical defect. Also shown is the line of maximum stress concentration associated with the mechanical defect

The horizontal dimension of the crack path was taken as the independent variable since the crack modeled (Fig. 1) was propagating from left to right in the horizontal direction. The vertical dimension of the crack path was taken as the dependent variable since the goal was to model the affect of composite architecture on displacements of the crack path in the vertical direction. A high affect of composite architecture would be predicted from significant displacements in the vertical direction, and a low affect would be predicted from no displacements in the vertical direction.

A crack path and the corresponding Langevin parameters must be understood in terms of the stress concentration around a mechanical defect [8]. If the material were completely uniform, the crack should propagate along the line of maximum stress concentration, i.e. the crack should propagate in a straight line from left to right in Fig. 1. If the crack propagation deviates from expected propagation path, some characteristic of the material must influence crack propagation. Two material characteristics that may influence crack paths in composite materials are material variability and geometric patterns. Material variability would result when individual material elements have stochastically distributed properties such as strength. A finite probability exists for a material element away from the maximum stress concentration to exceed its strength, but the probability decreases greatly as the distance from the maximum stress concentration increases. Therefore, material variability would be expected to result in a crack path that is generally from left to right, but wanders up and down slightly (Fig. 1) in response to stochastically distributed strength. The magnitude of the Langevin variability parameter, σ, would be expected to be related to material variability. If the magnitude of σ is 0, the material is completely uniform and failure of material elements at the maximum stress concentration would control the crack path. If the magnitude of σ is greater than 0, material variability exists and may influence the crack path.

Composites have a symmetry associated with the architecture of the reinforcing phase. The architecture is a description of the identity, amount, shape, and orientation of the reinforcing phase. If different architectures of the reinforcing phase result in dramatically different strengths, the architecture of the reinforcing phase could potentially provide alignment of strong and weak regions. As a result, the crack path may mirror the architecture of the reinforcing phase and deviate significantly from the crack path expected from the orientation of the maximum stress concentration. This phenomenon is similar to drift in a stochastic financial model, and can be thought of as architectural drift in a stochastic crack path model. The extent of architectural drift is defined by the magnitude of the Langevin drift parameter, μ. If the magnitude of μ is 0, there is no tendency for architectural drift in the crack path. It should be noted that μ can be either positive (architectural drift upward in Fig. 1) or negative (architectural drift downward in Fig. 1). A crack path in a composite would then be a function of the stress concentration, material variability (σ), and architectural drift (μ).

Figure 2 shows the affect of the Langevin variability parameter (σ) on predicted crack paths for the case of no architectural drift ($\mu = 0$). Figure 2a shows 10 predicted crack paths for the case of no architectural drift ($\mu = 0$) and moderate material variability ($\sigma = 0.05$). While the predicted crack paths were generally from left to right, they deviated up and down from a straight path expected for a non-stochastic crack path ($\mu = 0$ and $\sigma = 0$). The straight path is shown as a dashed line in Fig. 2a. Figure 2b shows 10 predicted crack paths for the case of no architectural drift ($\mu = 0$) and significant material variability ($\sigma = 0.1$). Again, the crack paths were predicted to be generally from left to right, but individual crack paths deviated significantly up and down from the straight path. The variability of the predicted crack paths clearly increased as the material variability (σ) increased.

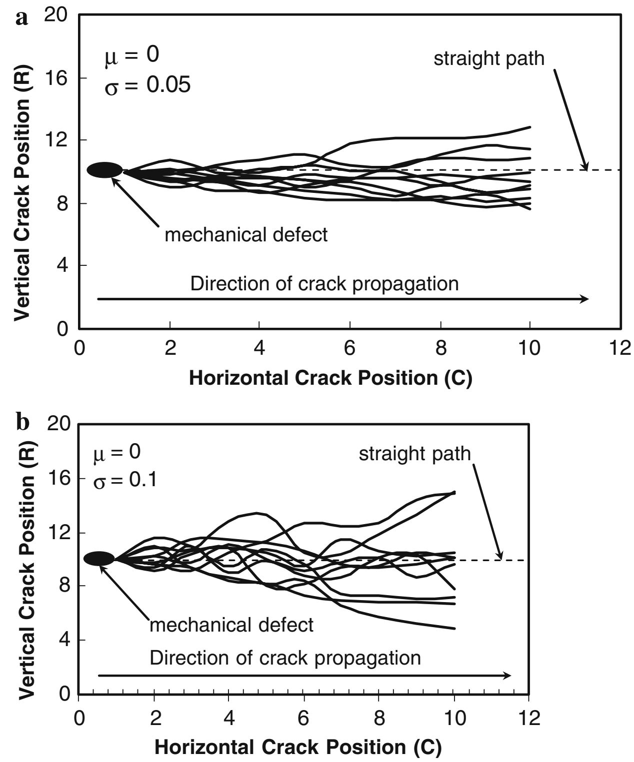

Fig. 2 Stochastic prediction of crack paths for a case of no architectural drift ($\mu = 0$) and (**a**) moderate material variability ($\sigma = 0.05$), and (**b**) significant material variability ($\sigma = 0.1$). Each plot contains 10 independently predicted crack paths. The non-stochastic crack path is shown as a dashed line, and is labeled straight path

Figure 3 shows the affect of the Langevin variability parameter (σ) on predicted crack paths for the case of moderate architectural drift ($\mu = 0.05$). Figure 3a shows 10 independent predictions for the case of no material variability ($\sigma = 0$). All 10 predictions indicated a single crack path moving upward away from the straight path. These predictions indicated that the crack is moving continuously away the line of maximum stress concentration, and would only be expected in conditions of the geometric alignment of weak regions. Such geometric alignment might potentially exist in composites with appropriate architectures.

Figure 3b shows 10 predicted crack paths for the case of moderate architectural drift ($\mu = 0.05$) and moderate material variability ($\sigma = 0.05$). The crack

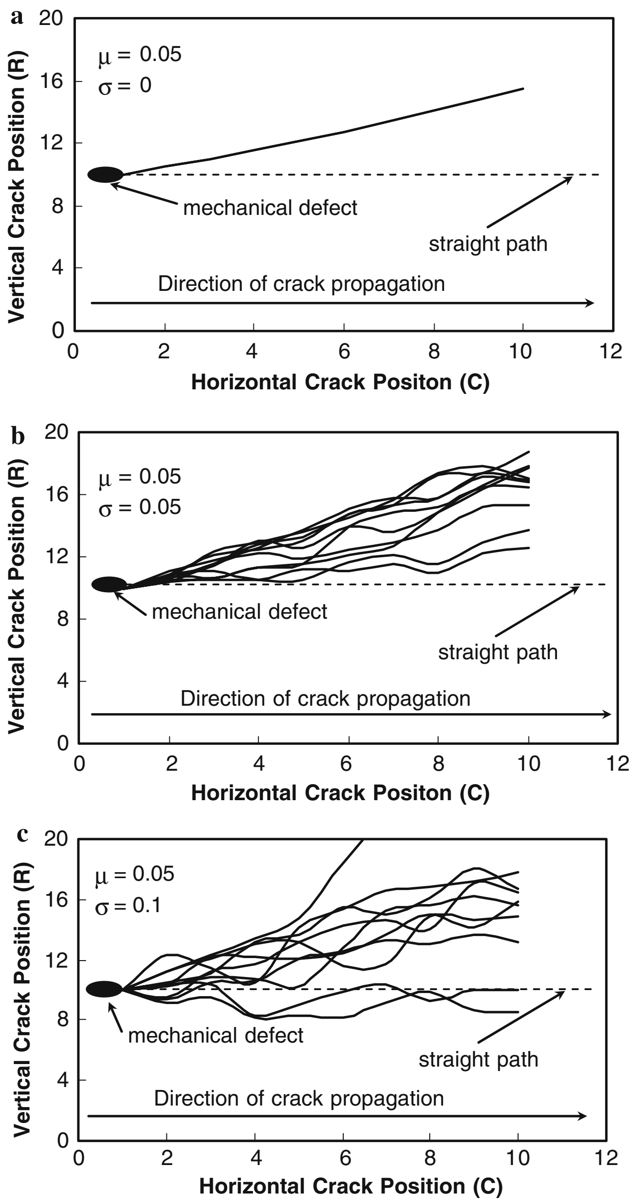

Fig. 3 Stochastic prediction of crack paths for a case of moderate architectural drift ($\mu = 0.05$) and (**a**) no material variability ($\sigma = 0$), (**b**) moderate material variability ($\sigma = 0.05$), and significant material variability ($\sigma = 0.1$). Each plot contains 10 independently predicted crack paths. The non-stochastic crack path is shown as a dashed line, and is labeled straight path

paths generally moved upward from left to right, but also exhibited moderate variation in position. Figure 3c shows 10 predicted crack paths for the case of moderate architectural drift ($\mu = 0.05$) and significant material variability ($\sigma = 0.1$). In this instance, individual crack paths exhibited significant variation. Some of crack paths were actually predicted to exist below the straight crack tip even though the architectural drift parameter suggested a path above the straight path.

Figure 4 shows the affect of the Langevin variability parameter (σ) on predicted crack paths for the case of significant architectural drift ($\mu = 0.1$). Figure 4a shows 10 predicted crack paths for the case of no material variability ($\sigma = 0$). All 10 predicted crack paths were the same, and the predicted path moved rapidly upward from the straight path. The predicted crack path was significantly different than the straight path,

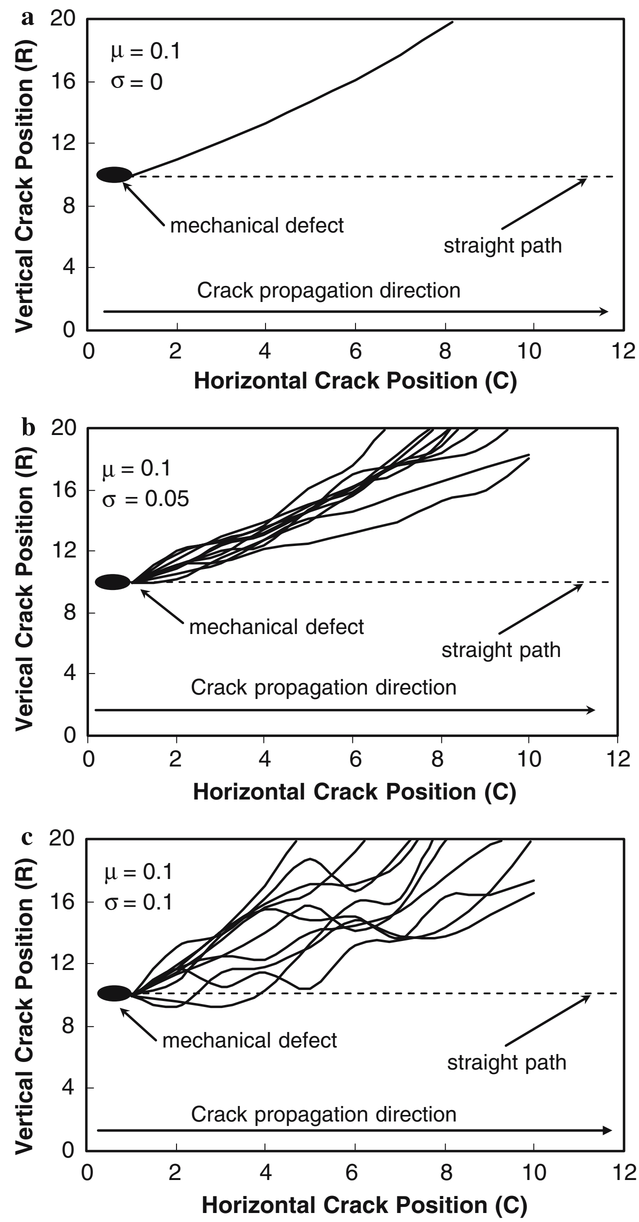

Fig. 4 Stochastic prediction of crack paths for a case of significant architectural drift ($\mu = 0.05$) and (**a**) no material variability ($\sigma = 0$), (**b**) moderate material variability ($\sigma = 0.05$), and significant material variability ($\sigma = 0.1$). Each plot contains 10 independently predicted crack paths. The non-stochastic crack path is shown as a dashed line, and is labeled straight path

and would only be expected under conditions of an alignment of weak regions in the material.

Figure 4b shows 10 predicted crack paths for the case of significant architectural drift (μ = 0.1) and moderate material variability (σ = 0.0.5). The general trend was a crack path moving from left to right and upward away from the straight path, but variation in individual crack paths was clearly evident. Figure 4c shows the case of significant architectural drift (μ = 0.1) and significant material variability (σ = 0.1). Again, the general trend was a crack path that moved upward away from the straight path, but significant variability was evident in individual crack paths.

Comparing Figures 2–4, two observations are clear. First, material variability (Langevin variability parameter σ) correlates strongly with the variation in a set of predicted crack paths. Higher material variability resulted in more variation in the predicted crack paths. Second, architectural drift (Langevin drift parameter μ) correlated strongly with the general orientation of the predicted crack path relative to the straight (non-stochastic) path. The combination of both the architectural drift (μ) and material variability (σ) provide a means to characterize position, shape, and variability of crack paths in a composite.

Figure 5a shows a crack path in a fiberglass reinforced epoxy matrix composite resulting from mechanical failure in bending. The composite specimen was 25 mm wide and 3 mm thick. The test geometry was simply supported, three-point bending with a support span of 125 mm. The specimen was monotonically loaded until failure of the tensile surface. The matrix phase was a brittle epoxy. The reinforcing fibers were comprised of bundles of about 50 glass fibers in each bundle. The diameter of a typical fiber was about 25 µm, the width of the fiber bundle was about 500 µm, and the length of the fiber bundles was about 15 mm for an aspect ratio of the bundle of about 30. The orientation of the fiber bundles was variable, but with a general orientation from the lower left to the upper right of Fig. 5a. The general orientation of fiber bundles is noted in Fig. 5a, and fiber bundles can be observed protruding out of the failure surface.

The crack path shown in Fig. 5a is represented as a two-dimensional trace in Fig. 5b. The direction of crack propagation was from left to right in both Fig. 5a and b. The initial crack path was essentially parallel to the general orientation of the fiber bundles, but then deviated from that direction to exhibit significant structure. The crack path shown in Fig. 5b was modeled with Eqs. 3 and 4 to yield the following Langevin parameters:

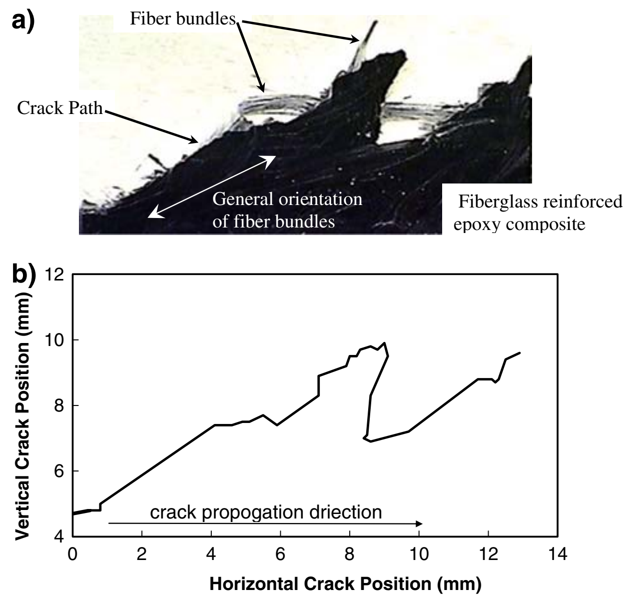

Fig. 5 (**a**) Experimentally determined crack path for a fiberglass reinforced epoxy composite tested in bending. The general orientation of the reinforcing phase is from the lower left to the upper right of the photograph, and the aspect ratio of the glass fiber bundles was approximately 30. (**b**) A two-dimensional trace of the crack path shown in **a**. The direction of crack propagation was from left to right

$$\mu = 0.04 \pm 0.02$$

$$\sigma = 0.05 \pm 0.03.$$

The Langevin parameters determined for the crack path shown in Fig. 5b are consistent with moderate architectural drift in the upward direction and moderate material variability. The moderate architectural drift Langevin parameter was consistent with the general orientation of reinforcing fiber bundles with aspect ratio 30. It should be noted that the relatively large uncertainties predicted in the Langevin parameters are related to the relatively large downward displacement of the crack path at about 9 mm horizontal crack position.

Figure 6a shows both the trace of the crack path in the composite (dark line) and 10 stochastic crack paths (dashed lines). The stochastic crack paths were predicted using Eq. 1, an architectural drift (μ) of 0.04, and a material variability (σ) of 0.05. While variation is evident in the stochastic crack paths shown in Fig. 6a, a strong correlation between the crack path in the fiberglass reinforced epoxy composite and one of the stochastic crack paths is clearly evident in Fig. 6b.

In summary, two general observations are evident. First, the architectural drift Langevin parameter predicted for a crack path in a fiberglass reinforced

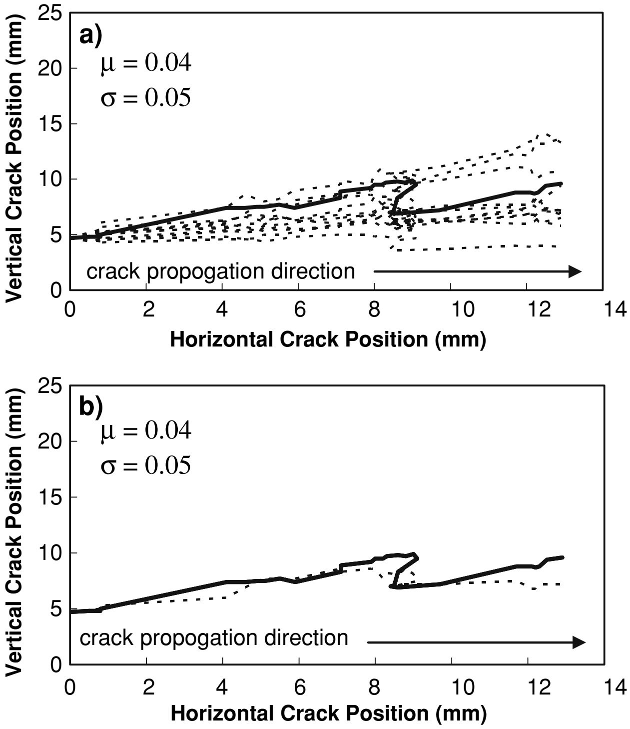

Fig. 6 (**a**) A comparison of the crack path in a fiberglass reinforced epoxy composite and 10 independent stochastic crack paths predicted using a Langevin model. The solid line represents the crack path observed in a fiberglass reinforced epoxy composite, and the dashed lines represent the predicted crack paths. (**b**) Crack path in a fiberglass reinforced epoxy composite and a single stochastic crack path showing a very strong correlation to the experimental crack path. The solid line represents the crack path observed in a fiberglass reinforced epoxy composite, and the dashed line represents a predicted crack path predicted with a Langevin model

epoxy is consistent with the geometry and orientation of the reinforcing phase. Second, there was a clear correlation between the experimentally determined crack path and the stochastic crack path predicting using a Langevin model. These two observations support the use of a stochastic Langevin model to quantify the affect of architecture of the reinforcing phase on crack paths in composite materials.

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